# Krivoliniski integral druge vrste (po koordinatama)

Ako je c data kriva u ravni opisana jednačinom $y=\eta(x)$ gdje je $a\le x\le b$ tada

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_a^b\left[P(x,\eta(x)+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

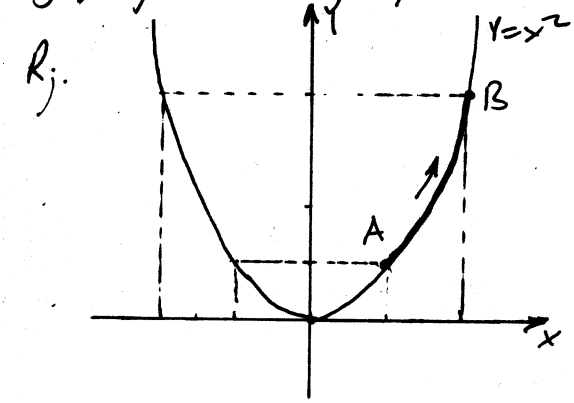
Ako je c data kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1\le t\le t_2$ tada

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_{t_1}^{t_2}\left[P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)\right]dt$$

Analogne formule vrijede za krivoliniski integral druge vrste uzete po prostornoj krivoj. Krivoliniski integral druge vrste OVISI O SMJERU PUTA INTEGRACIJE (bitna je orjentacija i u kom smjeru ide luk).

\# Izračunati krivoliniski integral $\int_c (x^2-2xy)\,dx+(2xy+y^2)\,dy$ gdje je c luk parabole $y=x^2$ od tačke $A(1,1)$ do $B(2,4)$.

Rj.




$y=x^2$

$\frac{dy}{dx}=2x$

$1\le x\le 2$

Ako je data kriva $y=\eta(x)$, $a\le x\le b$

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_a^b\left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

$$\int_c (x^2-2xy)\,dx+(2xy+y^2)\,dy=\int_1^2\left(x^2-2x^3+(2x^3+x^4)\,2x\right)dx=\int_1^2(2x^5+4x^4-2x^3+x^2)\,dx$$

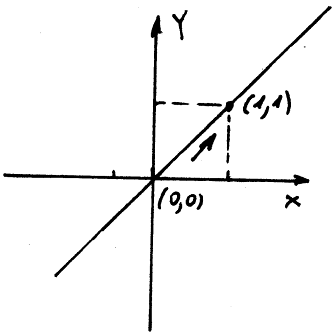
$$=2\cdot\frac{1}{6}x^6\Big|_1^2+4\cdot\frac{1}{5}x^5\Big|_1^2-2\cdot\frac{1}{4}x^4\Big|_1^2+\frac{1}{3}x^3\Big|_1^2=\frac{1}{3}\cdot 63+\frac{4}{5}\cdot 31-\frac{1}{2}\cdot 15+\frac{1}{3}\cdot 7=40+\frac{19}{30}$$

# Izračunati krivolinijski integral $\int\limits_{(0,0)}^{(1,1)} 2xy\,dx + x^2 dy$ ako prelazimo po liniji

a) $y = x$ b) $y = x^2$ c) $y = x^3$ d) $y^2 = x$.

Rj.

a)




$$\boxed{\text{Ako je data kriva } y = \eta(x),\ \alpha \le x \le \beta \quad \int\limits_{c} P(x,y)\,dx + Q(x,y)\,dy = \int\limits_{\alpha}^{\beta} [P(x,\eta(x)) + Q(x,\eta(x))\cdot \eta'(x)]\,dx}$$

$y = x$

$y' = 1$

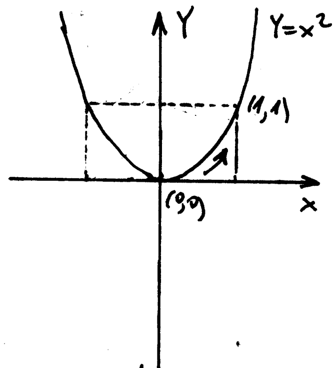
$$\int\limits_{(0,0)}^{(1,1)} 2xy\,dx + x^2 dy = \int\limits_0^1 (2x^2 + x^2\cdot 1)\,dx = 3\int\limits_0^1 x^2 dx = 3\cdot\frac{1}{3}x^3\Big|_0^1 = 3\cdot\frac{1}{3} = 1$$

b)

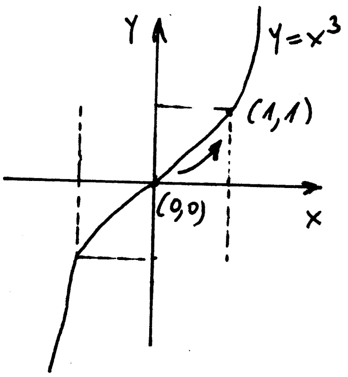
$y = x^2$

$y' = 2x$

$$\int\limits_{(0,0)}^{(1,1)} 2xy\,dx + x^2 dy = \int\limits_0^1 (2x\cdot x^2 + x^2\cdot 2x)\,dx = \int\limits_0^1 4x^3 dx = 4\cdot\frac{1}{4}x^4\Big|_0^1 = 4\cdot\frac{1}{4} = 1$$




c)




$y = x^3$

$y' = 3x^2$

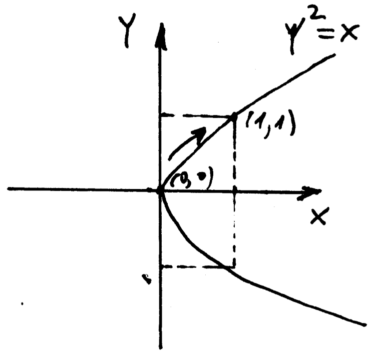
$$\int\limits_{(0,0)}^{(1,1)} 2xy\,dx + x^2 dy = \int\limits_0^1 (2x\cdot x^3 + x^2\cdot 3x^2)\,dx = \int\limits_0^1 5x^4 dx = 5\cdot\frac{1}{5}x^5\Big|_0^1 = 5\cdot\frac{1}{5} = 1$$

d)

$x = y^2$

$x'_y = 2y$

$$\int\limits_{(0,0)}^{(1,1)} 2xy\,dx + x^2 dy = \int\limits_0^1 (2\cdot y^2\cdot y\cdot 2y + (y^2)^2)\,dy = \int\limits_0^1 (4y^4 + y^4)\,dy = \int\limits_0^1 5y^4 dy = 5\cdot\frac{1}{5}y^5\Big|_0^1 = 1$$



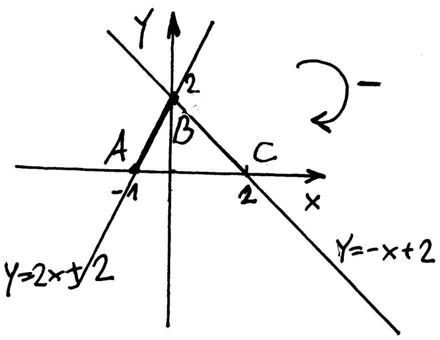
(#) Izračunati krivoliniske integrale

a) $\oint\limits_{-l} 2x\,dx - (x+2y)\,dy$

b) $\oint\limits_{+l} y\cos x\,dx + \sin x\,dy$

gdje je $l$ kontura trougla čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

Rj. a)
Nacrtajmo trougao $\Delta ABC$.




Provucimo pravu kroz tačke $B(\overset{x_1}{0};\overset{y_1}{2})$ i $C(\overset{x_2}{2};\overset{y_2}{0})$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

$$\frac{x}{2}=\frac{y-2}{-2} \quad /\cdot 2$$

$$x=-y+2$$

$$y=-x+2$$

Provucimo pravu kroz $A(\overset{x_1}{-1};\overset{y_1}{0})$ i $B(\overset{x_2}{0};\overset{y_2}{2})$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \qquad \frac{x+1}{1}=\frac{y}{2}$$

$$y=2x+2$$

$$\oint\limits_{-l} 2x\,dx-(x+2y)\,dy=\int\limits_{B(0;2)}^{C(2;0)} 2x\,dx-(x+2y)\,dy+\int\limits_{C(2;0)}^{A(-1;0)} 2x\,dx-(x+2y)\,dy+\int\limits_{A(-1;0)}^{B(0;2)} 2x\,dx-(x+2y)\,dy$$

$$\int\limits_{(0;2)}^{(2;0)} 2x\,dx-(x+2y)\,dy=\left|\begin{matrix} y=-x+2 \\ dy=-dx\end{matrix}\right|=\int\limits_{(0;2)}^{(2;0)}\left[2x-(x+2(-x+2))(-1)\right]dx=$$

$$=\int\limits_{(0;2)}^{(2;0)}\left[2x+x-2x+4\right]dx=\int\limits_{(0;2)}^{(2;0)}(x+4)\,dx=\left(\frac{1}{2}x^2+4x\right)\Big|_0^2=2+8=10$$

$$\int\limits_{C(2;0)}^{A(-1;0)} 2x\,dx-(x+2y)\,dy=\left|\begin{matrix} y=0 \\ dy=0\end{matrix}\right|=\int\limits_{2}^{-1}2x\,dx=2\cdot\frac{1}{2}x^2\Big|_2^{-1}=(1-4)=-3$$

$$\int\limits_{A(-1;0)}^{B(0;2)} 2x\,dx-(x+2y)\,dy=\left|\begin{matrix} y=2x+2 \\ dy=2\,dx\end{matrix}\right|=\int\limits_{-1}^{0}\left[2x-(x+2(2x+2))\,2\right]dx=$$

$$= \int_{-1}^{0} (2x - 2x - 8x - 8)\, dx = (-8) \int_{-1}^{0} (x+1)\, dx = (-8) \cdot \left[ \underbrace{\tfrac{1}{2} x^2 \Big|_{-1}^{0}}_{0-1} + \underbrace{x \Big|_{-1}^{0}}_{0+1} \right] =$$

$$= (-8) \cdot \left( -\tfrac{1}{2} + 1 \right) = -4$$

Prema tome $\displaystyle\oint_{\Delta ABC} 2x\,dx - (x+2y)\,dy = 10 - 3 - 4 = 3$

b) $\displaystyle\oint_{+\ell} y\cos x\,dx + \sin x\,dy = \int_{AC} y\cos x\,dx + \sin x\,dy + \int_{CB} y\cos x\,dx + \sin x\,dy$

$$+ \int_{BA} y\cos x\,dx + \sin x\,dy$$

$$\int_{A(-1;0)}^{C(2;0)} y\cos x\,dx + \sin x\,dy = \left| \begin{matrix} y = 0 \\ dy = 0 \end{matrix} \right| = \int_{-1}^{2} 0\,dx = 0$$

$$\int_{C(2;0)}^{B(0;2)} y\cos x\,dx + \sin x\,dy = \left| \begin{matrix} y = -x+2 \\ dy = -dx \end{matrix} \right| = \int_{2}^{0} \left[ (-x+2)\cos x - \sin x \right] dx$$

$$= \left| \begin{matrix} u = -x+2 & dv = \cos x \\ du = -1 & v = \sin x \end{matrix} \right| = (-x+2)\sin x \Big|_{2}^{0} + \int_{2}^{0} \sin x\,dx - \int_{2}^{0} \sin x\,dx = 0$$

$$\int_{B(0;2)}^{A(-1;0)} y\cos x\,dx + \sin x\,dy = \left| \begin{matrix} y = 2x+2 \\ dy = 2\,dx \end{matrix} \right| = \int_{0}^{-1} \left[ (2x+2)\cos x + 2\sin x \right] dx =$$

$$= 2 \int_{0}^{-1} \left[ (x+1)\cos x + \sin x \right] dx = \left| \begin{matrix} u = x+1 & dv = \cos x \\ du = dx & v = \sin x \end{matrix} \right| = 2(x+1)\sin x \Big|_{0}^{-1} - 2\int_{0}^{-1} \sin x\,dx$$

$$+ 2\int_{0}^{-1} \sin x\,dx = 0$$

Prema tome

$$\oint_{+\ell} y\cos x\,dx + \sin x\,dy = 0$$

# Izračunati integral $I=\oint\limits_C y^2\,dx$

po krivoj koja nastaje kao presjek kugle i valjka

$$x^2+y^2+z^2=R^2, \qquad x^2+y^2=Rx.$$

Rj.:

$$C:\begin{cases} x^2+y^2+z^2=R^2 \\ x^2+y^2=Rx \end{cases}$$

Posmatrajmo $xOy$ ravan.
Prvo napišimo krug $x^2+y^2=Rx$ u parametarskom obliku.

$$x^2+y^2=Rx$$

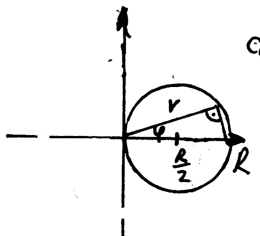
$$x^2-2\cdot x\cdot\frac{R}{2}+\frac{R^2}{4}-\frac{R^2}{4}+y^2=0$$

$$\left(x-\frac{R}{2}\right)^2+y^2=\left(\frac{R}{2}\right)^2$$

Prisjetimo se polarnih koordinata

$x=r\cos\varphi$

$y=r\sin\varphi$




$\cos\varphi=\dfrac{r}{R}$

$\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

U našem slučaju za krug $x^2+y^2=Rx$ za $r$ ćemo uzeti $r=R\cos\varphi$

Parametarski oblik kruga $x^2+y^2=Rx$ je

$x=R\cos\varphi\cos\varphi=R\cos^2\varphi$

$y=R\cos\varphi\sin\varphi$.

Uvrstimo ove vrijednosti u kuglu

$$x^2+y^2+z^2=R^2$$

$$R^2\cos^2\varphi\cos^2\varphi+R^2\cos^2\varphi\sin^2\varphi+z^2=R^2$$

$$R^2\cos^2\varphi+z^2=R^2$$

$$z^2=R^2-R^2\cos^2\varphi$$

$$z^2=R^2(1-\cos^2\varphi)$$

$$z^2=R^2\sin^2\varphi$$

Parametarski oblik date krive je:

$x=R\cos^2\varphi$

$y=R\cos\varphi\sin\varphi$

$z=R\sin\varphi$

$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$

$$I=\oint\limits_C y^2\,dx=\left|\begin{array}{l} x=R\cos^2\varphi \\ dx=R\,2\cos\varphi(-\sin\varphi)\,d\varphi \\ y=R\cos\varphi\sin\varphi \\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{array}\right|=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}R^2\cos^2\varphi\sin^2\varphi\cdot(-2)R\sin\varphi\cos\varphi\,d\varphi$$

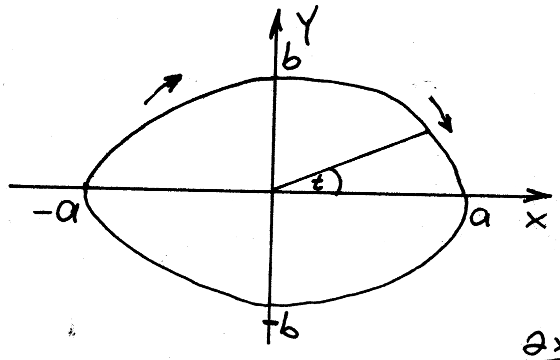
$$=(-2)R^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^3\varphi\cos^3\varphi\,d\varphi=(-2)R^3\cdot\frac{1}{8}\int_{-\pi/2}^{\pi/2}(2\sin\varphi\cos\varphi)^3\,d\varphi=-\frac{1}{4}R^3\cdot\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin2\varphi)^3\,d(2\varphi)$$

$$= -\frac{1}{8} R^3 \int_{-\frac{\pi}{2}}^{\pi/2} \sin^2 2\varphi \cdot \sin 2\varphi \, d(2\varphi) = -\frac{1}{8} R^3 \int_{-\frac{\pi}{2}}^{\pi/2} (1-\cos^2 2\varphi) \sin 2\varphi \, d(2\varphi) =$$

$$= +\frac{1}{8} R^3 \int_{-\frac{\pi}{2}}^{\pi/2} (1-\cos^2 2\varphi) \, d(\cos 2\varphi) = \frac{1}{8} R^3 \left( \underbrace{\cos 2\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{-1+1} - \frac{1}{3} \underbrace{\cos^3 2\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}}_{-1+1} \right) = 0$$

(#) Izračunati krivolinijski integral $\int\limits_c y^2 dx + x^2 dy$ gdje je c gornja polovina elipse $x = a\cos t$, $y = b\sin t$ $(a>0,\ b>0)$, koja se prelazi u smislu pomjeranja kazaljke na satu.

Rj.




Ako je kriva c zadana parametarski $x = \varphi(t)$, $y = \psi(t)$ gdje $\alpha \le t \le \beta$ imamo

$$\int\limits_c P(x,y)\,dx + Q(x,y)\,dy = \int\limits_\alpha^\beta \left[ P(\varphi(t),\psi(t))\varphi'(t) + Q(\varphi(t),\psi(t))\psi'(t) \right] dt$$

$$\frac{\partial x}{\partial t} = -a\sin t \qquad \frac{\partial y}{\partial t} = b\cos t$$

$$\int\limits_c y^2 dx + x^2 dy = \int\limits_\pi^0 \left[ b^2\sin^2 t \cdot (-a\sin t) + a^2\cos^2 t \cdot b\cos t \right] dt =$$

$$= -ab^2 \int\limits_\pi^0 \sin^3 t\, dt + a^2 b \int\limits_\pi^0 \cos^3 t\, dt \overset{(*)}{=} \frac{4}{3} a b^2$$

$$\int\limits_\pi^0 \sin^3 t\, dt = \int\limits_\pi^0 \sin t(1-\cos^2 t)\, dt = \left| \begin{matrix} \cos t = u & t=\pi \Rightarrow u=-1 \\ -\sin t\, dt = du & t=0 \Rightarrow u=1 \\ \sin t\, dt = -du & \end{matrix} \right| = -\int\limits_{-1}^1 (1-u^2)\, du =$$

$$= -\left( u\Big|_{-1}^1 - \frac{1}{3}u^3\Big|_{-1}^1 \right) = -\left(2 - \frac{1}{3}\cdot 2\right) = -\left(\frac{6-2}{3}\right) = -\frac{4}{3} \quad \ldots (*)$$

$$\int\limits_\pi^0 \cos^3 t\, dt = \int\limits_\pi^0 \cos t(1-\sin^2 t)\, dt = \left| \begin{matrix} \sin t = u \\ \cos t\, dt = du \\ t=\pi \Rightarrow u=0 \\ t=0 \Rightarrow u=0 \end{matrix} \right| = \int\limits_0^0 (1-u^2)\, du = 0$$

(#) Date su tačke $A(3;-6;0)$ i $B(-2;4;5)$. Izračunati krivolinijski integral $I=\int_c xy^2dx+yz^2dy-zx^2dz$ gdje je c:

a) duž koja spaja tačke O i B ( O koordinatni početak)
b) kriva od A do B: kruga zadan jednačinama
$x^2+y^2+z^2=45,\ 2x+y=0.$

Rj.
$$I=\int_c xy^2dx+yz^2dy+zx^2dz$$

Ovo je krivolinijski integral druge vrste. Prisjetimo se:
Ako je c kriva u prostoru opisana parametarskim jednačinama
$x=\mu(t),\ y=\eta(t),\ z=\theta(t)$ gdje je $t_1\le t\le t_2$ tada

$$\int_c P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz=\int_{t_1}^{t_2}P(\mu(t),\eta(t),\theta(t))\,\mu'(t)\,dt+$$
$$+\int_{t_1}^{t_2}Q(\mu(t),\eta(t),\theta(t))\,\eta'(t)\,dt+\int_{t_1}^{t_2}R(\mu(t),\eta(t),\theta(t))\,\theta'(t)\,dt$$

Da bi smo opisali duž $\overline{OB}$ u prostoru prvo postavimo pravu kroz ove dvije tačke.

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$ jednačina prave kroz dvije tačke $M_1(x_1,y_1,z_1)$ i $M_2(x_2,y_2,z_2)$

$O(0,0,0)$

$B(-2,4,5)$

$$\frac{x}{-2}=\frac{y}{4}=\frac{z}{5}\quad(=t)$$

$x=-2t$
$y=4t$
$z=5t$

Naše c je sada oblika
$$c:\begin{cases}x=-2t,\ y=4t,\ z=5t\\ 0<t<1\end{cases}$$

$$I=\int_c xy^2dx+yz^2dy-zx^2dz=\int_0^1\left((-2t)\,16t^2\cdot(-2)+4t\cdot 25t^2\cdot 4-5t\cdot 4t^2\cdot 5\right)dt=$$

$$=\int_0^1 (64t^3+400t^3-100t^3)\,dt = 364\int_0^1 t^3 dt = \frac{364}{4} = 91$$ traženo rješenje

b) Dat je krug u prostoru zadan jednačinama

$x^2+y^2+z^2=45$, $2x+y=0$

$\uparrow$ krug $\qquad \uparrow$ ravan

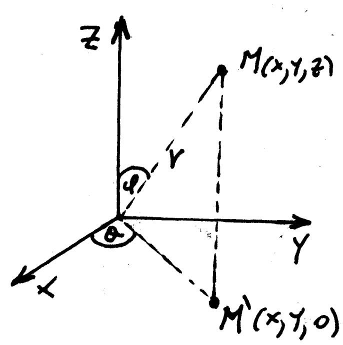
Da bi smo naš krug opisali u parametarskom obliku, veliku pomoć će odigrati sferne koordinate:

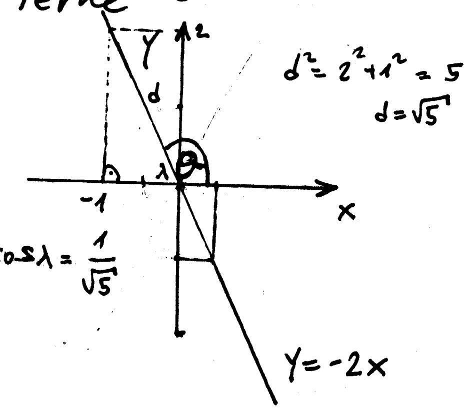
Sferne koordinate

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$




Da bi smo krug u prostoru opisali parametarski potrebno je u sfernim koordinatama fiksirati $r$ i $\theta$. U našem slučaju ugao $\theta$ nije moguće svesti na lijep oblik.

Pristupamo parametarizaciji kruga na drugi način:

$$\left.\begin{aligned} &2x+y=0 \Rightarrow y=-2x \\ &x^2+y^2+z^2=45 \Rightarrow z^2=45-x^2-y^2 \end{aligned}\right\} \Rightarrow c: \begin{cases} x=t \\ y=-2t \\ z=\sqrt{45-t^2-4t^2}=\sqrt{45-5t^2} \\ 3\le t\le -2 \end{cases}$$

$$dx = dt,\quad dy=-2dt,\quad dz=\frac{1}{2}(45-5t^2)^{-\frac{1}{2}}\cdot(-10t) = -\frac{5t}{\sqrt{45-5t^2}}\,dt$$

$$I=\int_c xy^2dx + yz^2dy - zx^2dz = \int_3^{-2}\left(t\cdot 4t^2 + (-2t)(45-5t^2)\cdot(-2) - \cancel{\sqrt{45-5t^2}}\cdot t^2\frac{(-5t)}{\cancel{\sqrt{45-5t^2}}}\right)dt$$

$$=\int_3^{-2}(4t^3+180t-20t^3+5t^3)\,dt = \int_3^{-2}(-11t^3+180t)\,dt$$

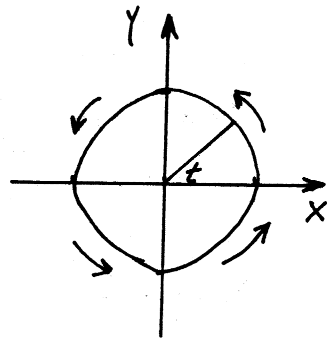
$$=-11\cdot\frac{1}{4}t^4\Big|_3^{-2} + 180\cdot\frac{1}{2}t^2\Big|_3^{-2} = -\frac{11}{4}\cdot(-65)+90\cdot(-5) = \frac{715-1800}{4} = \frac{-1085}{4}$$

$$=-271\frac{1}{4}$$ traženo rješenje

\# Izračunati krivoliniski integral $\int\limits_C \frac{(x+y)\,dx-(x-y)\,dy}{x^2+y^2}$

gdje je c krug $x^2+y^2=a^2$ koji se prelazi u smislu suprotnom pomjeranju kazaljke na satu.

Rj.




Krug $x^2+y^2=a^2$ napisan parametarski

$$x=a\cos t$$
$$y=a\sin t$$
$$0\le t\le 2\pi$$

Ako je c kriva zadana parametarski $x=\mu(t)$, $y=\eta(t)$, $a\le t\le b$

$$\int\limits_C P(x,y)\,dx+Q(x,y)\,dy = \int_a^b [P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)]\,dt$$

$$\frac{\partial x}{\partial t}=-a\sin t$$

$$\frac{\partial y}{\partial t}=a\cos t$$

$$\int\limits_C \frac{(x+y)\,dx-(x-y)\,dy}{x^2+y^2} = \int\limits_C \frac{x+y}{x^2+y^2}\,dx-\frac{x-y}{x^2+y^2}\,dy = \int_0^{2\pi}\Big[\frac{a\cos t+a\sin t}{a^2}\cdot(-a\sin t) -$$

$$-\frac{a\cos t-a\sin t}{a^2}\cdot a\cos t\Big]\,dt = \int_0^{2\pi}\big[(\cos t+\sin t)\cdot(-\sin t)-(\cos t-\sin t)\cdot\cos t\big]\,dt$$

$$=\int_0^{2\pi}(\underline{\underline{-\sin t\cos t}}-\sin^2 t-\cos^2 t\underline{\underline{+\sin t\cos t}})\,dt=\int_0^{2\pi}(-1)\,dt=-2\pi$$

(#) Izračunati krivolinijski integral $\int\limits_c x^3dx + 3zy^2dy - x^2y\,dz$ gdje je $c$ dio prave od tačke $A(3,2,1)$ do tačke $O(0,0,0)$.

Rj.

jednačina prave kroz dvije tačke $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$

$p(O,A)$: $\dfrac{x}{3} = \dfrac{y}{2} = \dfrac{z}{1} \quad (=t)$

$$\begin{cases} x=3t \\ y=2t \\ z=t \end{cases} \qquad \begin{aligned} dx &= 3\,dt \\ dy &= 2\,dt \\ dz &= dt \end{aligned}$$

Trebaju nam još granice za $t$

$A(3,2,1)$ $\quad \begin{aligned} x&=3t \\ y&=2t \\ z&=t \end{aligned} \quad \Rightarrow \quad t=1$

$O(0,0,0)$ $\quad \begin{aligned} x&=3t \\ y&=2t \\ z&=t \end{aligned} \quad \Rightarrow \quad t=0$

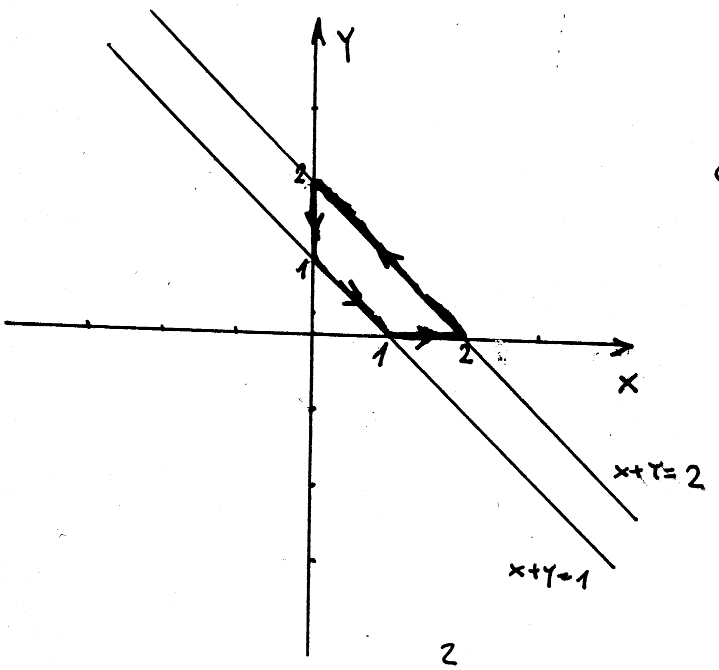
$$\int\limits_c x^2dx+3zy^2dy-x^2y\,dz =$$

$$= \int_1^0 \left[(3t)^3\cdot 3 + 3\cdot t\cdot(2t)^2\cdot 2 - (3t)^2\cdot 2t\cdot 1\right]dt$$

$$= \int_1^0 (81t^3+24t^3-18t^3)\,dt = -\int_0^1 87t^3\,dt = -87\cdot\frac{1}{4}t^4\Big|_0^1 = -\frac{87}{4}$$

(#) Izračunati krivoliniski integral $I=\int_C (x^2+y^2)\,dx + x^2y\,dy$ gdje je c kontura trapeza koja obrazuju prave $x=0,\ y=0,\ x+y=1,\ x+y=2$.

Rj.




Ako je $c:\ y=\eta(x),\ a\le x\le b$

$$\int_C P(x,y)\,dx+Q(x,y)\,dy = \int_a^b [P(x,\eta(x)) + Q(x,\eta(x))\cdot\eta'(x)]\,dx$$

U našem slučaju postoje 4 krive

$C_1:\ y=0,\ 1\le x\le 2$

$C_1:\ y=-x+2,\ 2\ge x\ge 0$

$C_3:\ x=0,\ 2\ge y\ge 1$

$C_4:\ y=-x+1,\ 0\le x\le 1$

$$I=I_1+I_2+I_3+I_4,\quad I_1=\int_1^2 (x^2+x^2\cdot 0)\,dx=\int_1^2 x^2\,dx=\frac{1}{3}x^3\Big|_1^2=\frac{1}{3}(8-1)=\frac{7}{3}$$

$$I_2=\int_2^0 \left(x^2+(-x+2)^2+x^2(-x+2)\cdot(-1)\right)dx=\int_2^0 (x^2+x^2-4x+4+x^3-2x^2)\,dx=$$

$$=\int_2^0 (x^3-4x+4)\,dx=\frac{1}{4}x^4\Big|_2^0-4\cdot\frac{1}{2}x^2\Big|_2^0+4x\Big|_2^0=-4+8-8=-4$$

$$I_3=\int_2^1 (y^2\cdot 0+0\cdot y)\,dy=0$$

$$I_4=\int_0^1 \left(x^2+(-x+1)^2+x^2(-x+1)\cdot(-1)\right)dx=\int_0^1 (x^2+x^2-2x+1+x^3-x^2)\,dx=$$

$$=\int_0^1 (x^3+x^2-2x+1)\,dx=\frac{1}{4}x^4\Big|_0^1+\frac{1}{3}x^3\Big|_0^1-2\cdot\frac{1}{2}x^2\Big|_0^1+x\Big|_0^1=\frac{1}{4}+\frac{1}{3}-1+1=\frac{7}{12}$$

$$I=I_1+I_2+I_3+I_4=\frac{7}{3}+(-4)+\frac{7}{12}=\frac{28-48+7}{12}=-\frac{13}{12}$$ vrijednost krivoliniskog integrala

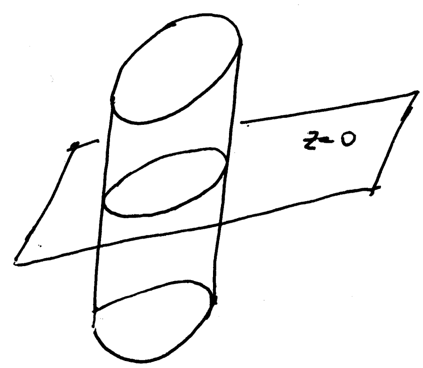
II način: Greenova formula ...

(#) Izračunati krivoliniski integral

$$I = \oint_C y\,dx + x^2\,dy$$

duž krive koja nastaje kao presjek ravni $z=0$ i cilindra $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x}{a}+\frac{y}{b}$ orjentisana u pozitivnom smjeru $(a \geq b > 0)$.

Rj. Za rješenje zadatka nije nam bitno gdje se cilindar nalazi u prostoru




$z=0$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x}{a}+\frac{y}{b}$$

$$\frac{1}{a^2}\left(x^2-ax\right)+\frac{1}{b^2}\left(y^2-by\right)=0$$

$$\frac{1}{a^2}\left(x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}\right)+\frac{1}{b^2}\left(y^2-2\cdot y\cdot\frac{b}{2}+\frac{b^2}{4}-\frac{b^2}{4}\right)=0$$

$$\frac{1}{a^2}\left(x-\frac{a}{2}\right)^2-\frac{1}{4}+\frac{1}{b^2}\left(y-\frac{b}{2}\right)^2-\frac{1}{4}=0$$

$$\frac{\left(x-\frac{a}{2}\right)^2}{a^2}+\frac{\left(y-\frac{b}{2}\right)^2}{b^2}=\frac{1}{2} \quad /\cdot 2$$

$$\frac{\left(x-\frac{a}{2}\right)^2}{\frac{a^2}{2}}+\frac{\left(y-\frac{b}{2}\right)^2}{\frac{b^2}{2}}=1$$

ovo je elipsa

Elipsu ćemo parametrizirati pomoću pomaknutih polarnih koordinata

$$x = \frac{a}{2} + \frac{a}{\sqrt{2}}\cos\varphi \qquad dx = -\frac{a}{\sqrt{2}}\sin\varphi \, d\varphi$$

$$y = \frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi \qquad dy = \frac{b}{\sqrt{2}}\cos\varphi \, d\varphi$$

$$0 \le \varphi < 2\pi$$

Sad nije teško izračunati dati krivolinijski integral

$$I = \oint_C y\,dx + x^2\,dy = \int_0^{2\pi} \left[ \left(\frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi\right) \frac{-a}{\sqrt{2}}\sin\varphi + \left(\frac{a}{2} + \frac{a}{\sqrt{2}}\cos\varphi\right)^2 \cdot \frac{b}{\sqrt{2}}\cos\varphi \right] d\varphi$$

$$= \frac{-a}{\sqrt{2}} \int_0^{2\pi} \left(\frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi\right)\sin\varphi \, d\varphi + \frac{b}{\sqrt{2}} \int_0^{2\pi} \left(\frac{a^2}{4} + \frac{a^2}{\sqrt{2}}\cos\varphi + \frac{a^2}{2}\cos^2\varphi\right)\cos\varphi \, d\varphi$$

$$= \frac{-ab}{2\sqrt{2}} \int_0^{2\pi} \sin\varphi \, d\varphi - \frac{ab}{2} \int_0^{2\pi} \underbrace{\sin^2\varphi}_{\frac{1}{2}(1-\cos 2\varphi)} d\varphi + \frac{a^2 b}{4\sqrt{2}} \int_0^{2\pi} \cos\varphi \, d\varphi + \frac{a^2 b}{2} \int_0^{2\pi} \underbrace{\cos^2\varphi}_{\frac{1}{2}(1+\cos 2\varphi)} d\varphi + \frac{a^2 b}{2\sqrt{2}} \int_0^{2\pi} \cos^3\varphi \, d\varphi$$

$$= \frac{-ab}{2\sqrt{2}} \underbrace{\cos\varphi \Big|_0^{2\pi}}_{=1-1=0} - \frac{ab}{2} \cdot \frac{1}{2}\left(\varphi\Big|_0^{2\pi} - \underbrace{\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}}_{=0}\right) + 0 + \frac{a^2 b}{2} \cdot \frac{1}{2}\left(\varphi\Big|_0^{2\pi} + \underbrace{\frac{1}{2}\sin 2\varphi\Big|_0^{4\pi}}_{=0}\right)$$

$$+ \frac{a^2 b}{2\sqrt{2}} \int_0^{2\pi} \cos^2\varphi \cos\varphi \, d\varphi =$$

$$= -\frac{ab\pi}{2} + \frac{a^2 b\pi}{2} + \frac{a^2 b}{2\sqrt{2}} \underbrace{\int_0^{2\pi} (1-\sin^2\varphi)\, d(\sin\varphi)}_{=0}$$

$$= \frac{\pi ab(-1+a)}{2} = \frac{ab\pi}{2}(a-1)$$

traženo rješenje

$$1 = \sin^2\varphi + \cos^2\varphi$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$$

$$1 - \cos 2\varphi = 2\sin^2\varphi$$

$$\sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi)$$

$$1 + \cos 2\varphi = 2\cos^2\varphi$$

II način: Greenova formula ...

(#) Izračunati krivolinijski integral

$$I = \oint_C z\,dz$$

duž krive koja nastaje kao presjek cilindra $\frac{(x-\frac{a}{2})^2}{\frac{a^2}{2}} + \frac{(y-\frac{b}{2})^2}{\frac{b^2}{2}} = 1$ i paraboloida $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ orjentisana u pozitivnom smjeru $(a \geq b > 0)$.

Rj. Prisjetimo se

Ako je kriva $C: \begin{cases} x = \mu(t) \\ y = \eta(t) \\ z = \theta(t) \\ t_1 \leq t \leq t_2 \end{cases}$ data u parametarskom obliku, tada

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \int_{t_1}^{t_2} \Big( P(\mu(t),\eta(t),\theta(t))\,\mu'(t) + Q(\mu(t),\eta(t),\theta(t))\,\eta'(t)$$
$$+ R(\mu(t),\eta(t),\theta(t))\,\theta'(t) \Big)\,dt$$

Da bi izračunali dati integral trebamo parametrizirati datu krivu. Uvrstimo smjene za $x$ i $y$ t.d. $\frac{(x-\frac{a}{2})^2}{\frac{a^2}{2}} + \frac{(y-\frac{b}{2})^2}{\frac{b^2}{2}} = 1$. $(*)$

Za $x$ i $y$ mogu nam pomoći poopštene polarne koordinate (gdje je $r$ fiksiran)

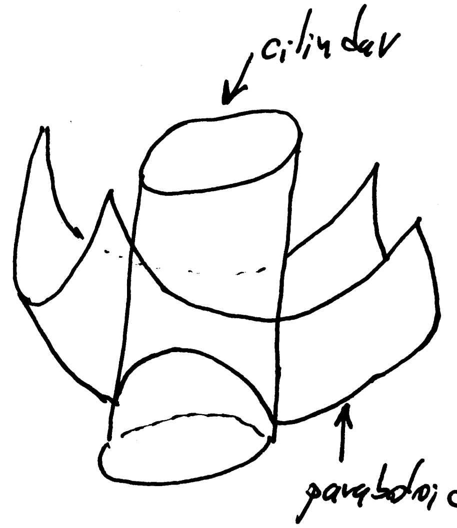
$$\left.\begin{array}{ll} x = \frac{a}{2} + \frac{a}{\sqrt{2}}\cos\varphi & \Rightarrow \left(x - \frac{a}{2}\right)^2 = \frac{a^2}{2}\cos^2\varphi \\ y = \frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi & \Rightarrow \left(y - \frac{b}{2}\right)^2 = \frac{b^2}{2}\sin^2\varphi \end{array}\right\} \Rightarrow \text{vrijedi } (*) \text{ za } \varphi \in [0, 2\pi)$$

Sada je

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\left(\frac{a}{2} + \frac{a}{\sqrt{2}}\cos\varphi\right)^2}{a^2} + \frac{\left(\frac{b}{2} + \frac{b}{\sqrt{2}}\sin\varphi\right)^2}{b^2} = \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\cos\varphi\right)^2 + \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\sin\varphi\right)^2 =$$

$$=\frac{1}{4}+\frac{1}{\sqrt{2}}\cos\varphi+\frac{1}{2}\cos^2\varphi+\frac{1}{4}+\frac{1}{\sqrt{2}}\sin\varphi+\frac{1}{2}\sin^2\varphi=$$

$$=1+\frac{1}{\sqrt{2}}\cos\varphi+\frac{1}{\sqrt{2}}\sin\varphi.$$




Prema tome imamo

$$z=1+\frac{1}{\sqrt{2}}\cos\varphi+\frac{1}{\sqrt{2}}\sin\varphi$$

$$dz=-\frac{1}{\sqrt{2}}\sin\varphi+\frac{1}{\sqrt{2}}\cos\varphi$$

$$\oint_c z\,dz=\int_0^{2\pi}\left(1+\frac{1}{\sqrt{2}}\cos\varphi+\frac{1}{\sqrt{2}}\sin\varphi\right)\left(-\frac{1}{\sqrt{2}}\sin\varphi+\frac{1}{\sqrt{2}}\cos\varphi\right)d\varphi=$$

$$=\int_0^{2\pi}\Big(-\frac{1}{\sqrt{2}}\sin\varphi+\frac{1}{\sqrt{2}}\cos\varphi\underline{-\frac{1}{2}\sin\varphi\cos\varphi}+\underbrace{\frac{1}{2}\cos^2\varphi-\frac{1}{2}\sin^2\varphi}_{=\frac{1}{2}(\cos 2\varphi-\sin 2\varphi)}\underline{+\frac{1}{2}\sin\varphi\cos\varphi}\Big)d\varphi$$

$$=-\frac{1}{\sqrt{2}}\int_0^{2\pi}\sin\varphi\,d\varphi+\frac{1}{\sqrt{2}}\int_0^{2\pi}\cos\varphi\,d\varphi+\frac{1}{2}\int_0^{2\pi}\cos 2\varphi\,d\varphi=$$

$$=-\frac{1}{\sqrt{2}}(-\cos\varphi)\Big|_0^{2\pi}+\frac{1}{\sqrt{2}}\sin\varphi\Big|_0^{2\pi}+\frac{1}{2}\cdot\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}=$$

$$=\frac{1}{\sqrt{2}}(1-1)+0+0=0$$

1. Izračunaj krivolinijski integral $I = \int\limits_L (xy-1)dx + x^2 y dy$ od tačke A(1,0) do tačke B(0,2).

a) po pravoj 2x+y=2

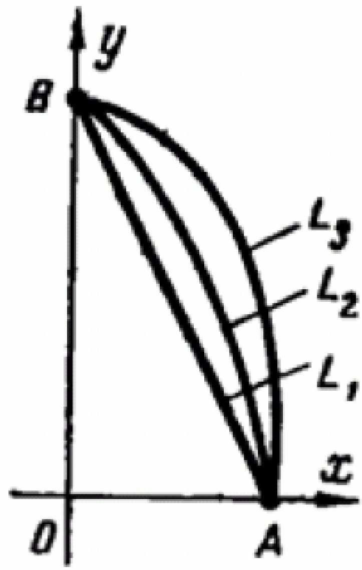
b) duž parabole $4x + y^2 = 4$

c) duž elipse x=cost ; y= 2sint

---

Rješenja:

a) Skicirajmo datu pravu (uputa vidi sliku desno).




2x+y=2
y = 2 - 2x
dy = -2dx

$$I = \int\limits_{L_1} (xy-1)dx + x^2 y dy =$$

$$= \int\limits_1^0 \left[x(2-2x)-1\right]dx + x^2(2-2x)(-2dx) =$$

$$= \int\limits_1^0 (2x-2x^2-1)dx + (-4x^2+4x^3)dx =$$

$$= \int\limits_1^0 (4x^3-6x^2+2x-1)dx = 4\cdot\frac{x^4}{4}\Big|_1^0 - 6\cdot\frac{x^3}{3}\Big|_1^0 + 2\cdot\frac{x^2}{2}\Big|_1^0 - x\Big|_1^0 = -1+2-1+1 = 1$$

---

b) Skicirajmo parabolu (uputa: vidi sliku iznad).

$$4x + y^2 = 4 \quad\Rightarrow\quad x = 1-\frac{y^2}{4} \quad\Rightarrow\quad dx = -\frac{y}{2}dy$$

$$I = \int\limits_{L_2} (xy-1)dx + x^2 y dy = \int\limits_0^2 \left[\left(1-\frac{y^2}{4}\right)y-1\right]\left(-\frac{y}{2}dy\right) + \left(1-\frac{y^2}{4}\right)^2 y dy =$$

$$= \int\limits_0^2 \left(y-\frac{y^3}{4}-1\right)\left(-\frac{y}{2}dy\right) + \left(1-\frac{y^2}{2}+\frac{y^4}{16}\right) y dy =$$

$$= \int\limits_0^2 \left(-\frac{y^2}{2}+\frac{y^4}{8}+\frac{y}{2}\right)dy + \left(y-\frac{y^3}{2}+\frac{y^5}{16}\right)dy =$$

$$=\int_0^2\left(\frac{y^5}{16}+\frac{y^4}{8}-\frac{y^3}{2}-\frac{y^2}{2}+\frac{3y}{2}\right)dy=\frac{y^6}{96}\Big|_0^2+\frac{y^5}{40}\Big|_0^2-\frac{y^4}{8}\Big|_0^2-\frac{y^3}{6}\Big|_0^2+\frac{3y^2}{4}\Big|_0^2=$$

$$=\frac{64}{96}+\frac{32}{40}-\frac{16}{8}-\frac{8}{6}+\frac{12}{4}=\frac{2}{3}+\frac{4}{5}-2-\frac{4}{3}+3=\frac{10+12-30-20+45}{15}=\frac{17}{15}.$$

---

c) Skicirajmo elipsu (uputa: vidi sliku sa prethodne stranice).

x = cost y = 2sint

dy = 2costdt

$$L_3:\begin{cases}x=\cos t\\ y=\sin t\\ 0\le t\le\dfrac{\pi}{2}\end{cases}$$

$$I=\int_{L_3}(xy-1)dx+x^2ydy=\int_0^{\frac{\pi}{2}}(\cos t\cdot 2\sin t-1)\cdot(-\sin tdt)+\cos^2 t\cdot 2\sin t\cdot 2\cos tdt=$$

$$=\int_0^{\frac{\pi}{2}}(-2\sin^2 t\cos t+\sin t)dt+4\cos^3 t\sin tdt=\int_0^{\frac{\pi}{2}}(4\cos^3 t\sin t+\sin t-2\sin^2 t\cos t)dt=$$

$$=4\int_0^{\frac{\pi}{2}}\cos^3 t\sin tdt+\int_0^{\frac{\pi}{2}}\sin tdt-2\int_0^{\frac{\pi}{2}}\sin^2 t\cos tdt\overset{*}{=}$$

$$\int\cos^3 t\sin tdt=\left|\begin{matrix}\cos t=u\\ -\sin tdt=du\\ \sin tdt=-du\end{matrix}\right|=-\int u^3du=-\frac{u^4}{4}+c=-\frac{\cos^4 t}{4}+c$$

$$\int\sin\ t\cos tdt=\left|\begin{matrix}\sin t=u\\ \cos tdt=du\end{matrix}\right|=\int u^2du=\frac{u^3}{3}+c=\frac{\sin^3 t}{3}+c$$

$$\overset{*}{=}4\cdot\left(-\frac{\cos^4 t}{4}\right)\Big|_0^{\frac{\pi}{2}}-\cos t\Big|_0^{\frac{\pi}{2}}-2\left(\frac{\sin^3 t}{3}\right)\Big|_0^{\frac{\pi}{2}}=4\cdot\frac{1}{4}+1-2\cdot\frac{1}{3}=2-\frac{2}{3}=\frac{4}{3}$$

---

# Zadaci za vježbu

U zadacima 3806 — 3821 izračunati date krivolinijske integrale.

**3806.** $\int\limits_L x\,dy$ po konturi trougla koji obrazuju koordinatne ose i prava $\frac{x}{2}+\frac{y}{3}=1$, — u pozitivnom smeru obilaženja (tj. nasuprot kretanju satne kazaljke).

**3807.** $\int\limits_L x\,dy$ po odsečku prave $\frac{x}{a}+\frac{y}{b}=1$, od tačke preseka sa apscisnom do tačke preseka sa ordinatnom osom.

**3808.** $\int\limits_L (x^2-y^2)\,dx$ po delu parabole $y=x^2$ od koordinatnog početka do tačke $(2, 4)$.

**3809.** $\int\limits_L (x^2+y^2)\,dy$ po konturi četvorougla čija su temena (navedena po redu obilaženja): $A(0, 0)$, $B(2, 0)$, $C(4, 4)$ i $D(0, 4)$.

**3810.** $\int\limits_{(0,0)}^{(\pi, 2\pi)} -x\cos y\,dx+y\sin x\,dy$ duž pravolinijskog odsečka koji spaja tačke $(0, 0)$ i $(\pi, 2\pi)$.

**3811.** $\int\limits_{(0,0)}^{(1,1)} xy\,dx+(y-x)\,dy$ duž krive 1) $y=x$, 2) $y=x^2$, 3) $y^2=x$, 4) $y=x^3$.

**3812.** $\int\limits_{(0,0)}^{(1,1)} 2xy\,dx+x^2\,dy$ duž krive 1) $y=x$, 2) $y=x^2$, 3) $y=x^2$, 4) $y^2=x$.

**3813.** $\int\limits_L y\,dx+x\,dy$ po delu kruga $x=R\cos t$, $y=R\sin t$, od $t_1=0$ do $t_2=\frac{\pi}{2}$.

**3814.** $\int\limits_L y\,dx-x\,dy$ po elipsi $x=a\cos t$, $y=b\sin t$, u pozitivnom smeru obilaženja.

**3815.** $\int\limits_L \frac{y^2\,dx-x^2\,dy}{x^2+y^2}$, po polukrugu $x=a\cos t$, $y=a\sin t$ od $t_1=0$ do $t_2=\pi$.

**3816.** $\int\limits_L (2a-y)\,dx-(a-y)\,dy$ duž prvog (računajući od koordinatnog početka) svoda cikloide $x=a(t-\sin t)$, $y=a(1-\cos t)$.

**3817.** $\int\limits_L \frac{x^2\,dy-y^2\,dx}{x^{\frac{5}{3}}+y^{\frac{5}{3}}}$, pri čemu je $L$ deo astroide $x=R\cos^3 t$, $y=R\sin^3 t$ od tačke $(R, 0)$ do tačke $(0, R)$.

**3818.** $\int\limits_L x\,dx+y\,dy+(x+y-1)\,dz$ duž pravolinijskog odsečka od tačke $(1, 1, 1)$ do tačke $(2, 3, 4)$.

**3819.** $\int\limits_L yz\,dx+z\sqrt{R^2-y^2}\,dy+xy\,dz$ po zavojnici $x=R\cos t$, $y=R\sin t$, $z=\frac{at}{2\pi}$, od njenog preseka sa ravni $z=0$ do preseka sa ravni $z=a$.

**3820.** $\int\limits_{(1,1,1)}^{(4,4,4)} \frac{x\,dx+y\,dy+z\,dz}{\sqrt{x^2+y^2+z^2-x-y+2z}}$ duž prave linije.

**3821.** $\int\limits_L y^2\,dx+z^2\,dy+x^2\,dz$ duž krive po kojoj se seku sfera $x^2+y^2+z^2=R^2$ i cilindar $x^2+y^2=Rx$ $(R>0,\ z\geqslant 0)$, pri čemu je smer obilaženja po konturi, posmatran iz koordinatnog početka, suprotan kretanju satne kazaljke.

# Rješenja

**3806.** 3. **3807.** $\frac{ab}{2}$.

**3808.** $-\frac{56}{15}$. **3809.** $37\frac{1}{3}$.

**3810.** $4\pi$. **3811.** 1) $\frac{1}{3}$; 2) $\frac{1}{12}$; 3) $\frac{17}{30}$; 4) $-\frac{1}{20}$.

**3812.** U sva četiri slučaja vrednost integrala je 1.

**3813.** 0. **3814.** $-2\pi ab$.

**3815.** $\frac{4}{3}a$. **3816.** $\pi a^2$.

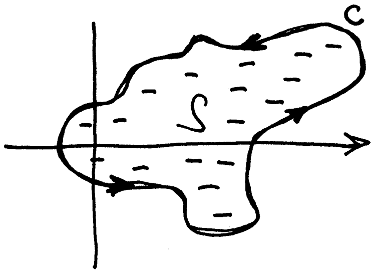
**3817.** $\frac{3}{16}\pi R\sqrt[3]{R}$

**3818.** 13. **3819.** $-\frac{a\pi R^4}{2}$

**3820.** $3\sqrt{3}$. **3821.** $-\frac{\pi R^3}{4}$.

# Greenova formula za ravan

Ako je $c$ po djelovima glatka granica područja $S$, a f-je $P(x,y)$ i $Q(x,y)$ neprekidne zajedno sa svojim parcijalnim izvodima prvog reda u zatvorenom području $S+C$, onda vrijedi Greenova formula




$$\int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

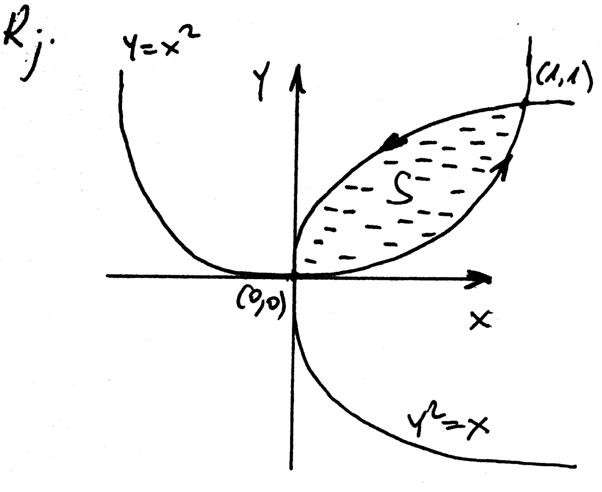
c - zatvorena kontura

S - oblast ograničena konturom

# Izračunati integral $\int_C (2xy - x^2)\,dx + (x+y^2)\,dx$

gdje je $c$ kontura površine ograničene sa $y=x^2$ i $y^2=x$.

Rj.




$P(x,y) = 2xy - x^2 \qquad \frac{\partial P}{\partial y} = 2x$

$Q(x,y) = x + y^2 \qquad \frac{\partial Q}{\partial x} = 1.$

$$\int_C P\,dx - Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

formula Greena

$$\int_C (2xy - x^2)\,dx + (x+y^2)\,dx = \iint_S (1-2x)\,dx\,dy = \int_0^1 \left[\int_{x^2}^{\sqrt{x}} (1-2x)\,dy\right] dx =$$

$$= \int_0^1 \left( y\Big|_{x^2}^{\sqrt{x}} - 2xy\Big|_{x^2}^{\sqrt{x}} \right) dx = \int_0^1 \left(\sqrt{x} - x^2 - 2x(\sqrt{x} - x^2)\right) dx =$$

$$= \int_0^1 \left(2x^3 - x^2 - 2x^{\frac{3}{2}} + x^{\frac{1}{2}}\right) = 2\cdot\frac{1}{4}x^4\Big|_0^1 - \frac{1}{2}x^3\Big|_0^1 - 2\cdot\frac{2}{5}x^{\frac{5}{2}}\Big|_0^1 + \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{1}{30}$$

(7) Izračunati krivoliniske integrale

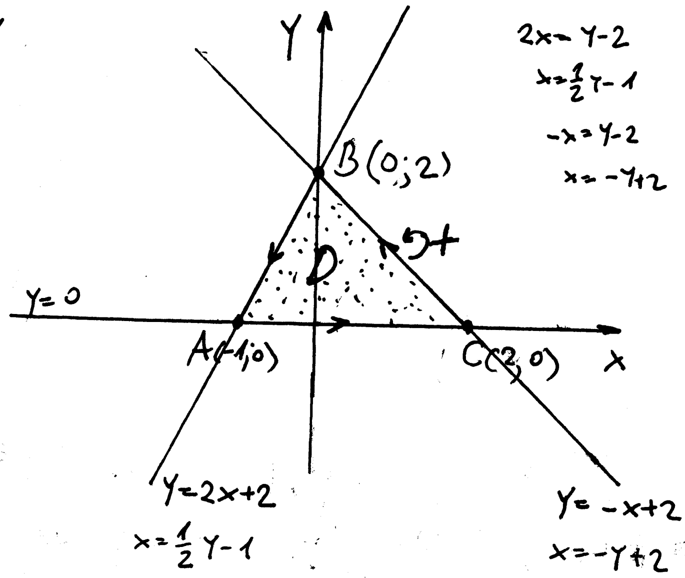
a) $\oint\limits_{-\ell} 2x\,dx-(x+2y)\,dy$ i b) $\oint\limits_{+\ell} y\cos x\,dx+\sin x\,dy$

po krivoj $\ell$, gdje je $\ell$ trougao čiji su vrhovi $A(-1;0)$, $B(0;2)$ i $C(2;0)$.

Rj.

$\int\limits_{c} P(x,y)\,dx+Q(x,y)\,dy$ je krivoliniski integral druge vrste.

Ako je kriva $c$ data u obliku $y=\eta(x)$, $c_1\le x\le c_2$ dati integral se računa po formuli
$$\int\limits_{c_1}^{c_2}\left(P(x,\eta(x))+Q(x,\eta(x))\,\eta'(x)\right)dx$$




Skicirajmo tačke u $xOy$ ravni

prava koja prolazi kroz tačke A i B je
$$\frac{x}{-1}+\frac{y}{2}=1 \quad /\cdot 2$$
$$-2x+y=1$$
$$y=2x+2 \quad \Rightarrow \quad y'=2$$

prava koja prolazi kroz tačke B i C
$$\frac{x}{2}+\frac{y}{2}=1 \quad /\cdot 2$$
$$x+y=2$$
$$y=-x+2 \quad \Rightarrow \quad y'=-1$$

a) $$\oint\limits_{-\ell} 2x\,dx-(x+2y)\,dy=\underset{\substack{AB\\ \text{po pravoj } y=2x+2}}{\int} 2x\,dx-(x+2y)\,dy+\underset{\substack{BC\\ \text{po pravoj } y=-x+2}}{\int} 2x\,dx-(x+2y)\,dy+\underset{\substack{CA\\ \text{po pravoj } y=0}}{\int} 2x\,dx-(x+2y)\,dy$$

$$\int\limits_{AB} 2x\,dx-(x+2y)\,dy=\int\limits_{-1}^{0}\Big[2x-\underbrace{(x+2(2x+2))\,2}_{\substack{(x+4x+4)\cdot 2\\ 10x+8}}\Big]dx=\int\limits_{-1}^{0}(-8x-8)\,dx=-8\cdot\frac{1}{2}x^2\Big|_{-1}^{0}-8x\Big|_{-1}^{0}$$
$$=(-4)(-1)-8=-4$$

$$\int\limits_{BC} 2x\,dx - (x+2y)\,dy = \int_0^2 \Big[2x - \underbrace{(x+2(-x+2))(-1)}_{\substack{(x-2x+4)(-1) \\ x-4}}\Big]dx = \int_0^2 (x+4)\,dx = \frac{1}{2}x^2\Big|_0^2 +$$

BC
po pravoj
$y=-x+2$

$$+4x\Big|_0^2 = 2+8=10$$

$$\int\limits_{CA} 2x\,dx - (x+2y)\,dy = \int_2^{-1} \big[2x - (x+2(0))0\big]\,dx = \int_2^{-1} 2x\,dx = 2\cdot\frac{1}{2}x^2\Big|_2^{-1} = 1-4=-3$$

CA
po pravoj
$y=0$

$$\oint\limits_{-\ell} 2x\,dx - (x+2y)\,dy = -4+10-3=3$$ traženo rješenje

b) Možemo upotrebiti Greenovu formulu

$$\int\limits_C P(x,y)\,dx + Q(x,y)\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$ gdje je $D$ oblast ograničena konturom $c$

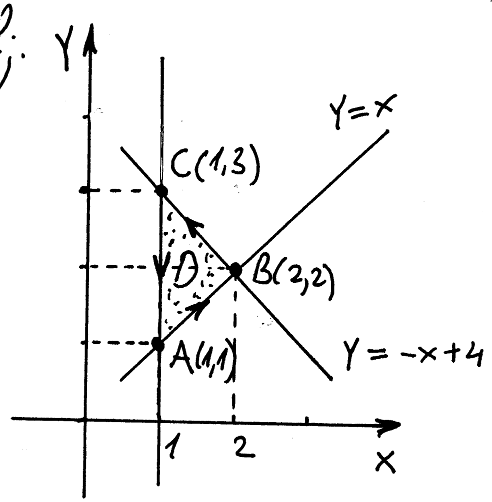
$$\oint\limits_{+\ell} y\cos x\,dx + \sin x\,dy = \left| \begin{array}{ll} Q(x,y) = \sin x & P(x,y) = y\cos x \\ \frac{\partial Q}{\partial x} = \cos x & \frac{\partial P}{\partial y} = \cos x \end{array} \right. $$

D-vidi sliku (tražkasti dio sa slike)

$$= \iint\limits_D (\cos x - \cos x)\,dx\,dy = \iint\limits_D 0\,dx\,dy = 0$$ traženo rješenje

(#) Izračunati $\int_c 2(x^2+y^2)dx+(x+y)^2dy$ gdje je c kontura trougla $\Delta ABC$ pozitivno orjentisana ($A(1,1)$, $B(2,2)$, $C(1,3)$).

Rj.




$$P(x,y) = 2(x^2+y^2) = 2x^2+2y^2$$

$$Q(x,y) = (x+y)^2 = x^2+2xy+y^2$$

$$\int_c P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

formula Grina

$$y-y_1 = \frac{x_2-x_1}{y_2-y_1}(x-x_1)$$

$$y-2 = \frac{-1}{1}(x-2)$$

$$y-2 = -x+2 \Rightarrow y=-x+4$$

$$\frac{\partial P}{\partial y} = 4y \qquad \frac{\partial Q}{\partial x} = 2x+2y \qquad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x+2y-4y = 2x-2y$$

$$D: \begin{cases} 1 \le x \le 2 \\ x \le y \le 4-x \end{cases} \qquad \int_c 2(x^2+y^2)dx + (x+y)^2dy = \iint_D (2x-2y)\,dx\,dy =$$

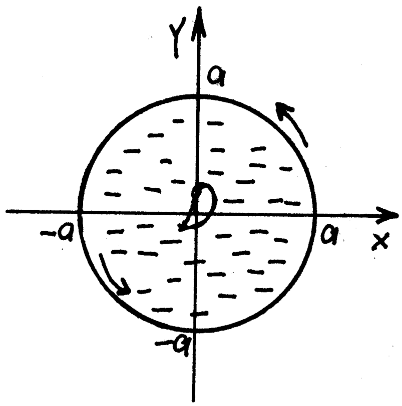
$$= \int_1^2 \left[\int_x^{4-x} (2x-2y)\,dy\right] dx = \int_1^2 \left(2xy\Big|_x^{4-x} - 2\cdot\frac{1}{2}y^2\Big|_x^{4-x}\right) dx =$$

$$= \int_1^2 \Big(2x(\overbrace{4-2x}^{4-x-x}) - (\overbrace{16-8x}^{(4-x)^2-x^2})\Big)\,dx = \int_1^2 (8x-4x^2-16+8x)\,dx = \int_1^2 (-4x^2+16x-16)\,dx$$

$$= -4\cdot\frac{1}{3}x^3\Big|_1^2 + 16\cdot\frac{1}{2}x^2\Big|_1^2 - 16x\Big|_1^2 = -\frac{4}{3}\cdot 7 + 8\cdot 3 - 16 = 8 - \frac{28}{3} = -\frac{4}{3}$$

(#) Izračunati $\int\limits_C xy^2\,dy - x^2y\,dx$ gdje je $c$ krug $x^2+y^2=a^2$. Integraciju izvesti u pozitivnom smjeru.

Rj.




$P(x,y) = -x^2y$ $\qquad \dfrac{\partial P}{\partial y} = -x^2$

$Q(x,y) = xy^2$ $\qquad \dfrac{\partial Q}{\partial x} = y^2$ $\qquad D: x^2+y^2 \le a^2$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 + x^2 = x^2 + y^2$$

$$\int\limits_C P(x,y)\,dx + Q(x,y)\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Formula Greena

polarne koordinate $\quad x = r\cos\varphi$, $\; y = r\sin\varphi \quad \Rightarrow \quad D': \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ 0 \le r \le a \\ 0 \le \varphi \le 2\pi \end{cases} \qquad dx\,dy = r\,dr\,d\varphi$

$$\int\limits_C xy^2\,dy - x^2y\,dx = \iint\limits_D (x^2+y^2)\,dx\,dy = \iint\limits_{D'} (r^2\cos^2\varphi + r^2\sin^2\varphi)\cdot r\,dr\,d\varphi =$$

$$= \int_0^{2\pi}\left[\int_0^a r^3\,dr\right] d\varphi = \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^a \, d\varphi = \frac{a^4}{4}\cdot \varphi\Big|_0^{2\pi} = \frac{\pi a^4}{2}$$

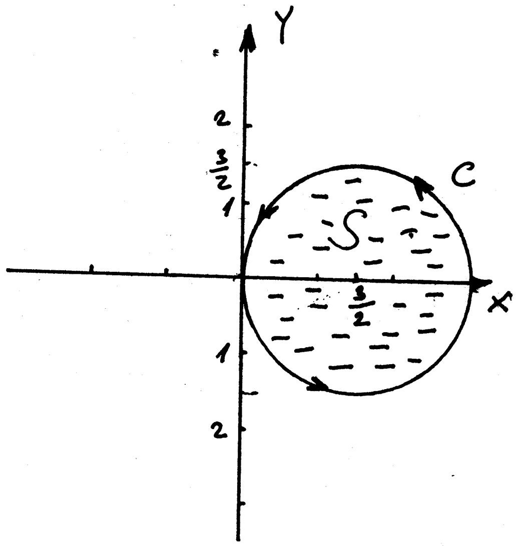
(#) Izračunati krivoliniski integral

$$I=\int_C (xy+x+y)\,dx+(xy+x-y)\,dy \quad \text{ako je } c: x^2+y^2=3x.$$

Rj.

$$x^2+y^2=3x$$
$$x^2-3x+y^2=0$$
$$x^2-2\cdot x\cdot\frac{3}{2}+\frac{9}{4}-\frac{9}{4}+y^2=0$$
$$\left(x-\frac{3}{2}\right)^2+y^2=\frac{9}{4}$$

c: Krug sa centrom u tački $\left(\frac{3}{2},0\right)$ poluprečnika $r=\frac{3}{2}$




I način: Greenova formula za ravan

$$\int_C P\,dx+Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

c - zatvorena kontura
S - oblast ograničena konturom

$P=xy+x+y$, $\frac{\partial P}{\partial y}=x+1$, $Q=xy+x-y$, $\frac{\partial Q}{\partial x}=y+1$.

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=y+1-(x+1)=y-x$$

Kako je c krug, oblast ograničena krugom je unutrašnjost kruga. Da bi smo lakše opisali unutrašnjost kruga uvedimo polarne koordinate

$$x=\frac{3}{2}+r\cos\varphi \qquad D:\begin{cases}0\le r\le \frac{3}{2}\\ 0\le\varphi\le 2\pi\end{cases}$$
$$y=r\sin\varphi$$
$$dx\,dy=r\,dr\,d\varphi$$

$$I=\int_C(xy+x+y)\,dx+(xy+x-y)\,dy=\iint_S(y-x)\,dx\,dy=\iint_D\left(r\sin\varphi-\left(\frac{3}{2}+r\cos\varphi\right)\right)\cdot r\,dr\,d\varphi$$

$$=\int_0^{3/2}\left[\int_0^{2\pi}\left(r^2\sin\varphi-\frac{3}{2}r-r^2\cos\varphi\right)d\varphi\right]dr=\int_0^{3/2}\left(\underbrace{-r^2\cos\varphi\Big|_0^{2\pi}}_{=0}-\frac{3r}{2}\varphi\Big|_0^{2\pi}\underbrace{-r\sin\varphi\Big|_0^{2\pi}}_{=0}\right)dr$$

$$=\int_0^{3/2}-3\pi r\,dr=-3\pi\frac{r^2}{2}\Big|_0^{3/2}=-\frac{3}{2}\pi\cdot\frac{9}{4}=-\frac{27}{8}\pi$$

II način: Klasičan način

$c$ kriva u ravni opisana jednačinom $y=\eta(x)$, $a\le x\le b$

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_a^b\left[P(x,\eta(x))+Q(x,\eta(x))\cdot\eta'(x)\right]dx$$

Ako je $c$ data kriva opisana parametarskim jednačinama $x=\mu(t)$, $y=\eta(t)$ gdje je $t_1\le t\le t_2$ tada

$$\int_c P(x,y)\,dx+Q(x,y)\,dy=\int_{t_1}^{t_2}\left[P(\mu(t),\eta(t))\mu'(t)+Q(\mu(t),\eta(t))\eta'(t)\right]dt$$

U našem slučaju $c$ je kružnica. Parametarizirajmo kružnicu

$x=\frac{3}{2}+r\cos\varphi$

$y=r\sin\varphi$

U našem slučaju $r=\frac{3}{2}$ a umjesto promjenljive $\varphi$ stavimo promjenljivu $t$

$$\frac{\partial x}{\partial t}=-\frac{3}{2}\sin t \qquad x=\frac{3}{2}+\frac{3}{2}\cos t$$

$$\frac{\partial y}{\partial t}=\frac{3}{2}\cos t \qquad y=\frac{3}{2}\sin t \qquad \text{gdje } 0\le t\le 2\pi$$

$$I=\int_c (xy+x+y)\,dx+(xy+x-y)\,dy=\int_0^{2\pi}\Big[\Big(\Big(\frac{3}{2}+\frac{3}{2}\cos t\Big)\Big(\frac{3}{2}\sin t\Big)+\Big(\frac{3}{2}+\frac{3}{2}\cos t\Big)+$$

$$+\Big(\frac{3}{2}\sin t\Big)\Big)\Big(-\frac{3}{2}\sin t\Big)+\Big(\Big(\frac{3}{2}+\frac{3}{2}\cos t\Big)\Big(\frac{3}{2}\sin t\Big)+\Big(\frac{3}{2}+\frac{3}{2}\cos t\Big)-$$

$$-\Big(\frac{3}{2}\sin t\Big)\Big)\frac{3}{2}\cos t\Big]dt=\ldots$$

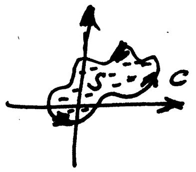
na klasičan način ovo je komplikovano ali se može izračunati

$$I=-\frac{27}{8}\pi$$

# Pomoću Greenove formule izračunati integral

$$I=\int_c (xy+x+y)\,dx+(xy+x-y)\,dy,$$ ako je $c$ kontura kružnice $x^2+y^2=ax$ pređena u pozitivnom smislu.

Rj: Greenova formula $$\int_c P\,dx+Q\,dy=\iint_S \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy.$$




$P(x,y)=xy+x+y$ $\qquad \dfrac{\partial P}{\partial y}=x+1,\quad \dfrac{\partial Q}{\partial x}=y+1$

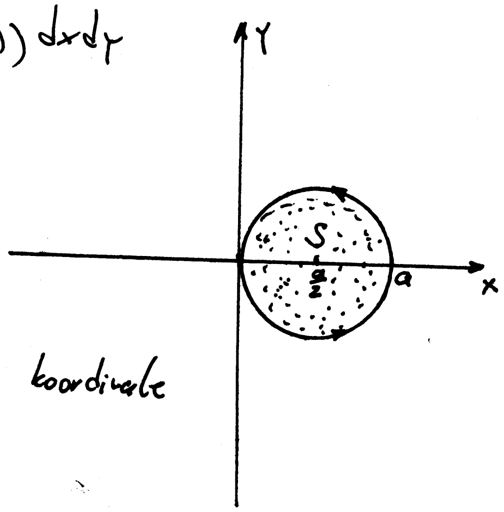
$Q(x,y)=xy+x-y$

$$x^2+y^2=ax$$
$$x^2-ax+y^2=0$$
$$x^2-2\cdot x\cdot\frac{a}{2}+\frac{a^2}{4}-\frac{a^2}{4}+y^2=0$$
$$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$

krug sa centrom u $\left(\frac{a}{2},0\right)$ poluprečnika $\frac{a}{2}$

$$I=\iint_S (y+1-(x+1))\,dx\,dy$$

$$I=\iint_S (y-x)\,dx\,dy$$




uvedimo polarne koordinate

$x=\frac{a}{2}+r\sin\varphi$

$y=r\cos\varphi$

$dx\,dy=r\,dr\,d\varphi$

$$S \xrightarrow{\text{transform.}} S': \begin{cases} 0\le r\le \frac{a}{2}\\ 0\le\varphi\le 2\pi\end{cases}$$

$$I=\iint_{S'}\left(r\cos\varphi-\frac{a}{2}+r\sin\varphi\right)r\,dr\,d\varphi=\iint_{S'}\left(r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right)dr\,d\varphi=$$

$$=\int_0^{2\pi}d\varphi\int_0^{\frac{a}{2}}\left[r^2(\cos\varphi-\sin\varphi)-\frac{a}{2}r\right]dr=\int_0^{2\pi}\left[\frac{1}{3}r^3\Big|_0^{\frac{a}{2}}(\cos\varphi-\sin\varphi)-\frac{a}{2}\cdot\frac{1}{2}r^2\Big|_0^{\frac{a}{2}}\right]d\varphi$$

$$=\frac{a^3}{24}\int_0^{2\pi}(\cos\varphi-\sin\varphi)\,d\varphi-\frac{a^3}{16}\int_0^{2\pi}d\varphi=\frac{a^3}{24}\Big(\underbrace{\sin\varphi\Big|_0^{2\pi}}_{=0}+\underbrace{\cos\varphi\Big|_0^{2\pi}}_{1-1}\Big)-\frac{a^3}{16}2\pi=$$

$$=-\frac{a^3\pi}{8}$$ traženo rješenje

# Zadaci za vježbu

U zadacima 3822—3823 krivolinijske integrale po zatvorenim konturama $L$, uzete u pozitivnom smeru obilaženja, transformisati u dvojne integrale po oblastima, ograničenim tim konturama.

**3822.** $\int\limits_L (1-x^2)\, y\, dx + x\,(1+y^2)\, dy.$

**3823.** $\int\limits_L (e^{xy}+2x\cos y)\, dx + (e^{xy}-x^2\sin y)\, dy.$

**3824.** Izračunati integral u zadatku 3822, ako je kontura integracije $L$ krug $x^2+y^2=R^2$, na dva načina:

1) neposredno;
2) primenom Grinove formule.

**3825.** Izračunati $\int\limits_L (xy+x+y)\, dx + (xy+x-y)\, dy$, pri čemu je kontura integracije $L$: 1) elipsa $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; 2) krug $x^2+y^2=ax$, a integral se uzima oba puta u pozitivnom smeru obilaženja. (Račun izvesti na dva načina: 1) neposredno, i 2) primenom Grinove formule).

**3826.** Dokazati da je integral $\int\limits_L (yx^3+e^y)\, dx + (xy^3+x\,e^y-2y)\, dy$ jednak nuli ako je putanja integracije $L$ zatvorena kriva simetrična u odnosu na koordinatni početak.

**3827.** Primenom Grinove formule izračunati razliku integrala

$$I_1=\int\limits_{AmB} (x+y)^2\, dx-(x-y)^2\, dy$$

i

$$I_2=\int\limits_{AnB} (x+y)^2\, dx-(x-y)^2\, dy,$$

pri čemu je $AmB$ pravolinijski odsečak koji spaja tačke $A(0,\,0)$ i $B(1,\,1)$, a $AnB$ je luk parabole $y=x^2$.

**3828.** Pokazati da je vrednost integrala $\int\limits_L \{x\cos(N,\,x)+y\sin(N,\,x)\}\, dS$, u kojem je $(N,\,x)$ ugao između spoljne normale krive $L$ i pozitivnog smera apscisne ose, uzetog u pozitivnom smeru obilaženja po zatvorenoj krivoj $L$, jednaka dvostrukoj površini oblasti ograničene zatvorenom krivom $L$.

**3829.** Dokazati da integral $\int\limits_L (2xy-y)\, dx + x^2\, dy$, uzet po zatvorenoj krivoj $L$, izražava površinu oblasti ograničene tom krivom.

**3830.** Dokazati da je integral $\int\limits_L \varphi(y)\, dx + [x\varphi'(y)+x^3]\, dy$ jednak trostrukom momentu inercije homogene ravne figure ograničene konturom $L$, u odnosu na ordinatnu osu.

# Rješenja

**3822.** $\iint\limits_D (x^2+y^2)\, dx\, dy.$

**3823.** $\iint\limits_D (y-x)\, e^{xy}\, dx\, dy.$

**3824.** $\frac{\pi R^4}{2}.$

**3825.** 1) 0; 2) $-\frac{\pi a^3}{8}.$

**3827.** $\frac{1}{3}.$